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# Vertex-magic total labelings of even complete graphs

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ABSTRACT

It is shown that if $p \geq 6$ is any even integer such that $p \equiv 2 \mathrm{mod}(4)$ then the complete graph $K_p$ has a vertex-magic total labeling (VMTL) with magic constant $h$ for each integer $h$ satisfying $p^3 + 6p \leq 4h \leq p^3 + 2p^2 - 2p$. If in addition, $p \equiv 2 \mathrm{mod}(8)$, then $K_p$ has a VMTL with magic constant $h$ for each integer $h$ satisfying $p^3 + 4p \leq 4h \leq p^3 + 2p^2$. If $p = 2 \cdot 3^t$ and $t \geq 2$, then it is shown that the complete graph $K_p$ has a VMTL with magic constant $h$ if and only if $h$ is an integer satisfying $p^3 + 3p \leq 4h \leq p^3 + 2p^2 + p$. These results provide significant new evidence supporting a conjecture of MacDougall, Miller, Slamin and Wallis regarding the spectrum of complete graphs. It is also shown that for each odd integer $n \geq 5$, the disjoint union of two copies of $K_n$, denoted $2K_n$, has a VMTL with magic constant $h$ for each integer $h$ such that $n^3+5n \leq 2h \leq n^3+2n^2-3n$. If in addition, $n \equiv 1 \mathrm{mod} 4$, then $2K_n$ has a VMTL with magic constant $h$ for each integer $h$ such that $n^3+3n \leq 2h \leq n^3+2n^2-n$.


## 1. Introduction

Let $G$ be a finite simple graph, with vertex set $V$ and edge set $E$. We will write $v = |V|$ and $e = |E|$ unless otherwise stated. Furthermore, all functions and variables in this paper take non-negative integer values unless otherwise noted.

A *labeling* $\lambda$ of $G$ is a map $\lambda : V \cup E \to \mathbb{Z}$. The *weight* $wt_\lambda(x)$ of a vertex $x$ is defined as $\lambda(x) + \sum \lambda(x, y)$ where the sum ranges over all edges incident with $x$. We say that $\lambda$ is *magic* if the weight $wt_\lambda(x)$ is a constant that does not depend on the choice of the vertex $x$. If $\lambda$ is magic we call the constant weight the *magic constant* of $\lambda$. A *vertex-magic total labeling* (or *VMTL*) $\tau$ is a magic labeling that is also a bijection $\tau : V \cup E \to \{1, 2, \ldots, v + e\}$. If the graph $G$ has at least one VMTL then $G$ is called a *vertex-magic* graph. The notion of a *feasible magic constant* was introduced in [8]. In order to define it we need the following standard calculation, essentially from [8].

**Proposition 1.** *Let $G = (V, E)$ be a graph and assume $G$ has a VMTL with magic constant $h$. Let $S_v$ denote the sum of all vertex labels and let $S_e$ denote the sum of all edge labels. Then*

1. $S_v + S_e = 1 + 2 + \cdots + (v + e)$.
2. $S_v + 2S_e = vh$.
3. $S_v = (v + e)(v + e + 1) - hv$.
4. $h = v + 2e + 1 + \left(e^2 + e - S_v\right)/v$.

Part (1) of the proposition is immediate from the definition. Part (2) follows from the fact that each edge is incident with two vertices and as a result its weight counts twice. Part (3) follows by using (1) and (2) to obtain an expression for $S_v$. Part (4) is an expression for $h$ obtained by rearranging part (3). One can use the obvious lower bound $S_v \geq 1 + 2 + \cdots + v$, together with part (4) of the proposition, to obtain an upper bound for $h$. Similarly one obtains a lower bound for $h$, using the fact that $S_e \geq 1 + 2 + \cdots + e$. The integral values between these bounds are called *feasible magic constants* for $G$.

The next proposition is also from [8]. The proof, which we omit, is straightforward.

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**Proposition 2.** *If $G$ is a regular graph of degree $d$, and $\tau$ is a VMTL with a magic constant of $h$, then the labeling that replaces each label $\tau(x)$ with $v+e+1-\tau(x)$ (called the dual of $\tau$) is also a VMTL. Furthermore, the dual of $\tau$ has a magic constant of $(d+1)(v+e+1)-h$.*

A VMTL is said to be *strong* if the smallest labels $1, 2, \ldots, e$ are used on the edges. A VMTL is *super* if the smallest labels $1, 2, \ldots, v$ are used on the vertices. If $G$ is regular, then the dual of a strong VMTL must therefore be a super VMTL, and vice versa. The *spectrum* of a graph $G$ is the set feasible values for which VMTLs of the graph actually exist. Vertex-magic total labelings were first defined by MacDougall et al. [8], who conjectured that for $p \geq 5$, the spectrum of $K_p$ coincides with the set of feasible magic constants for $K_p$. Thus:

**Conjecture 1** (*From [8]*). *Let $p \geq 5$ be an integer. Then $K_p$ has a VMTL with magic constant $h$ if and only if $h$ is an integer satisfying the inequality $(p/4)\left(p^2+3\right) \leq h \leq (p/4)\,(p+1)^2$.*

It is a trivial exercise to show that $K_2$ is *not* vertex-magic. Computer searches [8] show that the above conjecture holds for $p = 5, 6$, but that it does not hold for $p = 4$, as three feasible values for $K_4$, namely 19, 22 and 25, are not in the spectrum of $K_4$. In [8] it is shown that each *odd order* complete graph has *at least one* VMTL. Lin and Miller [7] proved that if $p \equiv 0 \bmod (4)$, then $K_p$ is vertex-magic. In [5] a VMTL was constructed for all *even order* complete graphs other than $K_2$. Part of the strategy in that paper was to find a VMTL for $2K_p$ and then add $p^2$ edges to get $K_{2p}$—the added edges would be labeled by shifting the numbers from a $p \times p$ magic square. This will be a big part of our strategy as well. Indeed, we construct many new VMTLs for $2K_n$, for odd $n \geq 5$, in Section 2, and then use them in Section 3 to obtain VMTLs for $K_{2n}$ with many different magic constants (Theorems 8 and 9). The graph $2K_3$ is the only regular graph of degree at least 2 which is known *not to be vertex-magic*. In Section 4 there will be further discussion of this point, as well as a discussion of several related open problems. Conjecture 1 was also mentioned by Gomez in [2]. His paper showed that $K_{4t}$ has a super VMTL for $t \geq 2$, resolving a conjecture in [9] that such labelings exist. It follows from an application of Proposition 1 part (4), and it is noted in [2], that any super VMTL of $K_p$ must have a magic constant of $(p/4)(p+1)^2$, which is the largest feasible constant. The question of which feasible values are in the spectrum of $K_p$ for $p \equiv 0 \bmod (4)$ remains open.

There are two papers (other than this one) that classify the spectra of infinite families of regular graphs of degree at least 2. We use the results of both of those papers in this work. In [12], the second author and Smith proved that Conjecture 1 does indeed hold for all *odd* $p$, but the techniques and proofs in that paper rely crucially on the assumption that $p$ is odd. These labelings will be used in Section 2 to obtain labelings of $2K_p$. In [11] the second author and McQuillan showed that if $s = 2 \cdot 3^t$, $t \geq 1$, then the spectrum of the graph $sK_3$ coincides with its feasible magic constants (where $sK_3$ denotes the disjoint union of $s$ copies of $K_3$). Edges are added to obtain a complete graph with $3s$ vertices, and these edges are labeled using a result of Shiu et al. [13]. The result is the solution to the spectrum problem (Theorem 14) for $K_s$, $s = 2 \cdot 3^t$, for each $t \geq 2$.

## 2. The spectrum of $2K_n$, for odd $n \geq 5$

Throughout this section we assume that $n$ is *odd* with $n = 2m-1$. The main goal of this section is to prove Theorems 3 and 5, by providing VMTLs with almost all feasible values for $2K_n$, $n \geq 5$.

**Definition.** If $\alpha : V \cup E \to \{0, 1, \ldots, s-1\}$ is an onto labeling, then we say that $\alpha$ is an *s-surjection*. If in addition, the labeling $\alpha$ is a magic labeling, then we say that $\alpha$ is a magic *s-surjection*.

**Proposition 3.** *If $G$ is a $d$-regular graph with $v$ vertices and $dv/2$ edges then $\alpha$ is a magic $(v+dv/2)$-surjection with magic constant $h$ if and only if $\alpha+1$ is a VMTL with magic constant $h+d+1$.*

Thus, in order to find VMTLs for $K_n$, it will be enough to find magic $mn$-surjections, since $K_n$ has $mn$ vertices and edges. The following proposition summarizes a few facts proven as part of the construction in [12]:

**Proposition 4.** *Let $r$ be an integer such that $0 \leq r \leq n-1$, and let $h = n\left(m^2-1\right)-r$. Then $K_n$ has a magic $mn$-surjection $\delta$ with magic constant $h$, and a corresponding magic $m$-surjection $\mu$ such that:*

1. *For each vertex $x$, $\mu(x) = 0$ and $0 \leq \delta(x) < 2n$.*
2. *For each edge $e$, $\mu(e) = 1$ if and only if $0 \leq \delta(e) < 2n$.*
3. *For each $i \geq 2$ and for each edge $e$, $\mu(e) = i$ if and only if $n\mu(e) \leq \delta(e) < (n+1)\mu(e)$.*
4. *Each vertex is incident with exactly 2 edges that $\mu$ labels with $i$, for each $1 \leq i \leq m-1$.*

An example, corresponding to $n = 5$ and $r = 2$, is shown in Fig. 1. For this particular example, we construct a magic $2mn$-surjection for $2K_n$ as follows. Set $a_0 = 0, a_1 = 3, a_2 = 4$ and set $b_0 = 2, b_1 = 1, b_2 = 5$. For the first component, label each object $z$ with $\delta(z) + (a_i - \mu(z))n$ if and only if $\mu(z) = i$. For the second component, label each object $z$ with $\delta(z) + (b_i - \mu(z))n$ if and only if $\mu(z) = i$. The new labeling, shown in Fig. 2, is evidently magic. An easy calculation shows that the magic constant is the same on both components, essentially due to the fact that $b_0 - a_0 = 2\left((a_1+a_2)-(b_1+b_2)\right)$, together with part 4 of the previous proposition. Due to the special treatment of edges that $\mu$ labels with '1' in the above proposition, we chose $b_1$ and $a_1$ so that $\{a_1, b_1\} = \{a_0+1, b_0+1\}$. For this small example with $n = 5$, we could not choose $a_1 = a_0 + 1$, however it will be possible for $n \geq 9$. This motivates the following:

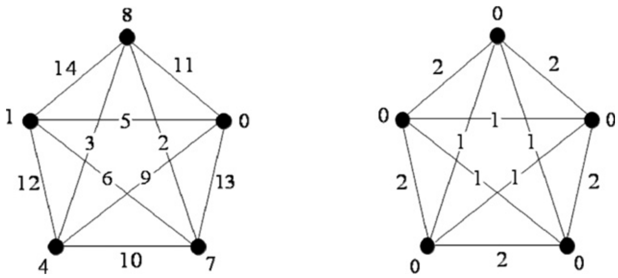


**Fig. 1.** A magic 15-surjection $\delta$ for $K_5$ (left) and a corresponding magic 3-surjection $\mu$ for $K_5$ (right).

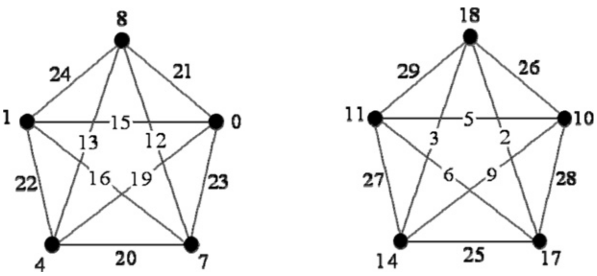


**Fig. 2.** A magic 30-surjection for $2K_5$.

**Definition.** A $(p, q)$-partition of a set $\{0, 1, \ldots, 2m-1\}$ is a partition of that set into two parts $A$ and $B$ such that

1. $|A| = |B|$.
2. $p, p+1 \in A$.
3. $p+2q, p+2q+1 \in B$. (In particular, $p+2q+1 \leq 2m-1$.)
4. $\sum_B - \sum_A = q$.

**Remark.** Since $\sum_{i=0}^{2m-1} i = (2m-1)m$, it follows from the definition above that $\sum_A = \frac{(2m-1)m-q}{2}$. We will use this fact in the proof of the next theorem.

First we show how to use such a partition to construct a magic $2mn$-surjection (and hence a VMTL) for $2K_n$, and then for $n \geq 9$, we construct enough of these partitions to obtain VMTLs with many different magic constants.

**Theorem 1.** *Let $n = 2m-1$, $m \geq 3$ and assume that $\{0, 1, \ldots, n\}$ has a $(p, q)$-partition. Then for each $r = 0, 1, \ldots, n-1$, there is a magic $2mn$-surjection of $2K_n$ with magic constant $n\left(2m^2 - 1 - q - p\right) - r$.*

**Proof.** As in the example, we make good use the magic $mn$-surjection $\delta$ of $K_n$ from Proposition 4. Let $A = \{a_0, \ldots, a_{m-1}\}$ with $a_0 = p$. For the first component, label each object $z$ with $\delta(z) + (a_i - \mu(z))n$ if and only if $\mu(z) = i$. Each vertex will therefore have a weight of:

$$wt_\delta(v) + a_0 n + 2n \sum_{i=1}^{m-1} (a_i - i) = wt_\delta(v) - pn + 2n \sum_A -2n \sum_{i=1}^{m-1} i.$$

Let $B = \{b_0, \ldots, b_{m-1}\}$ with $b_0 = p + 2q$. For the second component, label each object $z$ with $\delta(z) + (b_i - \mu(z))n$ if and only if $\mu(z) = i$. Each vertex of this component will, similarly, have a weight of:

$$wt_\delta(v) - (p+2q)n + 2n \sum_B -2n \sum_{i=1}^{m-1} i.$$

Since $q = \sum_B - \sum_A$ it follows that the weight of each vertex in the first component is equal to the weight of each vertex in the second component. Thus we have a magic labeling with a constant of:

$$\begin{aligned} wt_\delta(v) - pn + 2n \sum_A -2n \sum_{i=1}^{m-1} i &= n(m^2-1) - r - pn + n\left[(2m-1)m - q\right] - n(m-1)m \\ &= n(2m^2 - 1 - p - q) - r. \end{aligned}$$

The result follows. $\square$

Our next task is to make good use of Theorem 1, by constructing suitable $(p, q)$-partitions. We first consider the case where $n \equiv 1 \bmod 4$, $n \geq 9$.

**Theorem 2.** *For each odd $m \geq 5$, the set $\{0, 1, \ldots, 2m-1\}$ has a $(p, 1)$-partition for each $p = 0, 1, \ldots, 2m-4$.*

**Proof.** For each $n = 2m-1$, and each $p$, we will define a set $A_p^n$. In each case the reader can set $B = \{0, 1, \ldots, 2m-1\} - A_p^n$ and check that the pair $A_p^n, B$ does yield a $(p, 1)$-partition. The proof is by induction on $n$. First assume $n = 9$. For $p = 0, 1, \ldots, 6$ we have the following choices for $A_p^9$:

| $p$: | 0 | 1 | 2 | 3 | 4 | 5 | 6 |
|---|---|---|---|---|---|---|---|
| $A_p^9$: | $\{0, 1, 6, 7, 8\}$ | $\{1, 2, 5, 6, 8\}$ | $\{2, 3, 1, 7, 9\}$ | $\{3, 4, 0, 7, 8\}$ | $\{4, 5, 1, 3, 9\}$ | $\{5, 6, 0, 2, 9\}$ | $\{6, 7, 1, 3, 5\}$ |

Now assume $n \geq 13$ and that the result holds for $n-4$. If $p = 0, 1, \ldots, n-7$, then set $A_p^n = A_p^{n-4} \cup \{n-3, n\}$.
If $n-6 \leq p \leq n-3$ then set $A_p^n = \{0, 3\} \cup \left\{x + 4 | x \in A_{p-4}^{n-4}\right\}$. $\square$

**Proposition 5.** *The disjoint union $2K_5$ has a VMTL with magic constant $h$ whenever $70 \leq h \leq 85$.*

**Proof Sketch.** A series of ad hoc constructions, many similar to the one in the example that resulted in Fig. 2, were used to provide magic 30-surjections for $2K_5$ with magic constants 73, 74, ..., 80. By Proposition 3 there are VMTLs with magic constants 78, 79, ..., 85. Duality (Proposition 2) yields VMTLs with magic constants 70, 71, ..., 77. $\square$

**Theorem 3.** *Assume that $n \geq 5$ and $n \equiv 1 \bmod 4$. Then there is a VMTL for $2K_n$ with magic constant $h$ for each integer $h$ satisfying $(1/2)\left(n^3 + 3n\right) \leq h \leq (1/2)\left(n^3 + 2n^2 - n\right)$.*

**Proof.** By Proposition 5 we will assume that $n \geq 9$. By Theorem 2, $\{0, 1, \ldots, n\}$ has a $(p, 1)$-partition for each $p = 0, 1, \ldots, n-3$, and so we may apply Theorem 1 to obtain a magic $2mn$-surjection of $2K_n$ with magic constant $x$ whenever $n\left(2m^2 - 2 - (n-3)\right) - (n-1) \leq x \leq n(2m^2 - 2)$. By Proposition 3, $2K_n$ has a VMTL with a magic constant of $h$ whenever $n\left(2m^2 - 2 - (n-3)\right) + 1 \leq h \leq n(2m^2 - 1)$, or equivalently, whenever $(1/2)\left(n^3 + 3n\right) + 1 \leq h \leq (1/2)\left(n^3 + 2n^2 - n\right)$. It remains to show that there is a VMTL with a magic constant of exactly $(1/2)\left(n^3 + 3n\right)$. But this now follows from duality. Indeed, we have just shown that there is a VMTL with a magic constant of $h = (1/2)\left(n^3 + 2n^2 - n\right)$, and so by Proposition 2 there is a VMTL with a magic constant of $n\left(2n + n(n-1) + 1\right) - h = (1/2)\left(n^3 + 3n\right)$, which completes the proof. $\square$

**Theorem 4.** *For each even $m \geq 6$, the set $\{0, 1, \ldots, 2m-1\}$ has a $(p, 2)$-partition for each $p = 0, 1, \ldots, 2m-6$.*

**Proof.** As in the proof of Theorem 2, we will define a set $A_p^n$, and the reader can easily check that $A_p^n$, together with its complement, form the required $(p, 2)$-partition. The proof is by induction on $n$. First assume $n = 11$. For $p = 0, 1, \ldots, 6$ we have the following choices for $A_p^{11}$:

| $p$: | 0 or 1 | 2 | 3 | 4 | 5 or 6 |
|---|---|---|---|---|---|
| $A_p^{11}$: | $\{0, 1, 2, 8, 10, 11\}$ | $\{2, 3, 4, 5, 8, 10\}$ | $\{3, 4, 0, 5, 9, 11\}$ | $\{4, 5, 0, 2, 10, 11\}$ | $\{5, 6, 7, 2, 4, 8\}$ |

Now assume $n \geq 15$ and that the result holds for $n-4$. Thus, for each $p = 0, 1, \ldots, n-9$, set $A_p^n = A_p^{n-4} \cup \{n-3, n\}$. If $n-8 \leq p \leq n-5$ then set $A_p^n = \{0, 3\} \cup \left\{x + 4 | x \in A_{i-4}^{n-4}\right\}$. $\square$

If $n = 7$, we could not find suitable $(p, 2)$-partitions, however, through ad hoc methods, we were able to find VMTLs as described in the following:

**Proposition 6.** *The disjoint union $2K_7$ has a VMTL with magic constant $h$ whenever $189 \leq h \leq 210$.*

**Theorem 5.** *Assume that $n \geq 7$ and $n \equiv 3 \bmod 4$. Then there is a VMTL for $2K_n$ with magic constant $h$ for each integer $h$ satisfying $(1/2)\left(n^3 + 5n\right) \leq h \leq (1/2)\left(n^3 + 2n^2 - 3n\right)$.*

**Proof.** By Proposition 6, we will assume that $n \geq 11$. By combining Theorem 1 with Theorem 4, we see that there is a magic $2mn$-surjection of $2K_n$ with magic constant $x$ whenever $n\left(2m^2 - 3 - (n-5)\right) - (n-1) \leq x \leq n(2m^2 - 3)$. By Proposition 3, the graph $2K_n$ has a VMTL with a magic constant of $h$ whenever $n\left(2m^2 - 3 - (n-5)\right) + 1 \leq h \leq n(2m^2 - 2)$, or equivalently, whenever $(1/2)\left(n^3 + 5n\right) + 1 \leq h \leq (1/2)\left(n^3 + 2n^2 - 3n\right)$. It remains to show that there is a VMTL with a magic constant of exactly $(1/2)\left(n^3 + 5n\right)$. But this now follows from Proposition 2 using the (just constructed) VMTL with magic constant $h = (1/2)\left(n^3 + 2n^2 - 3n\right)$. $\square$

**Remark.** It follows from Proposition 1 part 4 that the range of feasible values for the graph $2K_n$ consists of the *integers* between $(1/2)\,n^3 + n$ and $(1/2)\,n^3 + n^2$. Hence we have shown, in Theorem 3, that at least $n^2 - 2n + 1$ of the $n^2 - n$ feasible values for $2K_n$ are in its spectrum, for $n \geq 5$ and $n \equiv 1 \bmod 4$. Likewise, by Theorem 5 we have shown that at least $n^2 - 4n + 1$ of the $n^2 - n$ feasible values for $2K_n$ are in the spectrum if $n \geq 7$ and $n \equiv 3 \bmod 4$.

## 3. Vertex-magic total labelings of complete graphs of order $2n$, for odd $n \geq 5$.

In this section we will use our VMTLs for $2K_n$ to construct VMTLs for the even complete graph $K_{2n}$. Furthermore, if $s \equiv 2 \bmod 4$ and $s \geq 6$, we will use VMTLs for $sK_3$ to provide VMTLs for the even complete graph $K_{3s}$. First we need a few definitions and well-known facts, which can be found in [16].

Let $G = (V, E)$ be a graph. A bijective map $\gamma : E \to \{1, 2, \ldots, |E|\}$ is called a *vertex-magic edge labeling* (sometimes called a *supermagic* labeling) if for each vertex $v$, the sum of the edge labels on edges incident with $v$ is a constant that does not depend on $v$. It is well known that for each $n \neq 2$ there is an $n \times n$ square $M = [m_{i,j}]$ with entries $1, 2, \ldots, n^2$ with a constant row sum of $(1/2)\,n\left(n^2+1\right)$ and a constant column sum of $(1/2)\,n\left(n^2+1\right)$. (If one further imposes the condition that the forward diagonal sum and the backward diagonal sum are also $(1/2)\,n\left(n^2+1\right)$, then $M$ is called a *magic square.*) Given the complete bipartite graph $K_{n,n}$ with vertex sets $V_1 = \{x_1, \ldots, x_n\}$ and $V_2 = \{y_1, \ldots, y_n\}$ the labeling that assigns $m_{i,j}$ to the edge $(x_i, y_j)$ is easily seen to be a vertex-magic edge labeling of $K_{n,n}$. Now assume $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ are disjoint graphs. The *join* of $G_1$ and $G_2$, denoted $G_1 \vee G_2$, is the union of $G_1$, $G_2$ and the complete bipartite graph with vertex sets $V_1$ and $V_2$. The next theorem summarizes a discussion from [16] page 87.

**Theorem 6.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order $n$ and assume that $\alpha$ is a VMTL for the disjoint union $G_1 \cup G_2$, with magic constant $h$. Then the graph $G_1 \vee G_2$ has a VMTL $\beta$ with magic constant $h + n\,(2n + |E_1| + |E_2|) + (1/2)\,n\left(n^2+1\right)$.*

**Proof Sketch.** If $z$ is a vertex or edge of $G_1 \cup G_2$ then set $\beta(z) = \alpha(z)$. If $x_i \in V_1$ and $y_j \in V_2$ then set $\beta(x_i, y_j) = 2n + |E_1| + |E_2| + m_{i,j}$. □

A *Latin square* with entries $1, 2, \ldots, n$ is an $n \times n$ array such that each number $1, 2, \ldots, n$ appears once in each row, and each column. Two Latin squares $A = [a_{i,j}]$ and $B = [b_{i,j}]$ are *orthogonal* if the $n^2$ ordered pairs $(a_{i,j}, b_{i,j})$ formed by superimposing one square on the other, are different. As a result, the square $C = [c_{i,j}]$ defined by $c_{i,j} = a_{i,j} + n(b_{i,j} - 1)$ would have a constant row sum, and a constant column sum of $(1/2)\,n\left(n^2+1\right)$. It is known [15,16] that for each $n$ other than 1, 2 or 6, there does exist a pair of orthogonal $n \times n$ Latin squares. If the magic square used in the construction of $\beta$ in Theorem 6 is obtained by combining orthogonal Latin squares in this way, then we can get additional VMTLs by shifting all labels of $G_1 \cup G_2$ up and shifting some other labels down.

**Theorem 7.** *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be graphs of order $n \neq 1, 2, 6$. Assume that $\alpha$ is a VMTL for the disjoint union $G_1 \cup G_2$, with magic constant $h$. Then $G_1 \vee G_2$ has a VMTL $\beta_k$ with magic constant $h + n\,(2n + |E_1| + |E_2| - k) + (1/2)\,n\left(n^2+1\right)$ for each $k = 0, 1, \ldots, n$.*

**Proof.** If $z$ is a vertex or edge of $G_1 \cup G_2$ then set $\beta_k(z) = \alpha(z) + kn$. Let $C = [c_{i,j}]$ be as in the preceding discussion. If $x_i \in V_1$ and $y_j \in V_2$ then set $\beta_k(x_i, y_j) = d_{i,j}$, where

$$d_{i,j} = \begin{cases} c_{i,j}, & \text{if } 1 \leq b_{i,j} \leq k \\ c_{i,j} + 2n + |E_1| + |E_2|, & \text{if } k+1 \leq b_{i,j} \leq n. \end{cases}$$

It is easy to check that $\beta_k$ is a VMTL. To calculate the magic constant for $\beta_k$, compare it to the magic constant for $\beta_0$ from Theorem 6, using Proposition 1 part 4. Only one term in this proposition, namely "$-S_v/v$" actually depends on the choice of VMTL for a fixed graph. Since each vertex label has increased by $kn$, we see that the constant for $\beta_k$ must be $kn$ less than the constant for $\beta_0$. The result follows. □

**Corollary 1.** *Assume $n \geq 5$ is odd and that $2K_n$ has a VMTL with magic constant $h$. Then the even complete graph $K_{2n}$ has a VMTL with magic constant $h + \frac{3n^3+n}{2} + n^2 - kn$, for each $k = 0, 1, \ldots, n$.*

**Proof.** This is immediate from Theorem 7, setting $G_1 \cong G_2 \cong K_n$, and therefore, $|E_1| = |E_2| = (1/2)(n^2 - n)$. □

**Theorem 8.** *Assume that $n \equiv 1 \bmod 4$ and that $n \geq 5$. Then the even complete graph $K_{2n}$ has a VMTL with magic constant $h$ for each integer $h$ such that $2n^3 + 2n \leq h \leq 2n^3 + 2n^2$.*

**Proof.** By Corollary 1, each magic constant $h$ for $2K_n$ yields $n+1$ different magic constants for $K_{2n}$, where these magic constants for $K_{2n}$ form an arithmetic progression with common difference $n$. By Theorem 3, we can choose any value for $h$ such that $(1/2)\left(n^3+3n\right) \leq h \leq (1/2)\left(n^3+2n^2-n\right)$. Since $n \geq 5$, this set of possible values for $h$ consists of (more than $n$) consecutive integers, and so the total set of magic constants one can obtain by combining Theorem 3 and Corollary 1 must consist of a string of consecutive integers. The smallest such integer (obtained by setting $k = n$) must be $(1/2)(n^3 + 3n) + (1/2)(3n^3 + n) = 2n^3 + 2n$ and the largest such integer (obtained by setting $k = 0$) must be $(1/2)(n^3 + 2n^2 - n) + (1/2)(3n^3 + n) + n^2 = 2n^3 + 2n^2$. □

**Theorem 9.** *If $n \equiv 3 \bmod 4$ then the even complete graph $K_{2n}$ has a VMTL with magic constant $h$ for each integer $h$ such that $2n^3 + 3n \leq h \leq 2n^3 + 2n^2 - n$.*

**Remark.** The proof is almost the same as the proof of Theorem 8, and therefore, we omit some of the discussion.

**Proof.** As mentioned in the introduction, the case where $n = 3$ was dealt with in [8]. Therefore we will assume that $n \geq 7$. We combine Corollary 1 with Theorem 5. The smallest magic constant we can obtain in this manner is $(1/2)(n^3 + 5n) + (1/2)(3n^3 + n)$, and the largest is $(1/2)(n^3 + 2n^2 - 3n) + n^2$. A straightforward enumeration shows that each integer in between these extremes can also be obtained as a magic constant for $K_{2n}$. $\square$

**Corollary 2.** *If $n \geq 3$ is odd then the even complete graph $K_{2n}$ has a VMTL with magic constant $h$ for each integer $h$ such that $2n^3 + 3n \leq h \leq 2n^3 + 2n^2 - n$.*

**Proof.** This is immediate from Theorems 8 and 9. $\square$

Note that by setting $p = 2n$ in Corollary 2 (respectively in Theorem 8) one obtains the result claimed in the abstract for VMTLs for $K_p$ when $p \equiv 2 \bmod 4$ (respectively when $p \equiv 2 \bmod 8$).

Our final goal is to solve the spectrum problem for $K_{2n}$ where $n = 3^k$ for each $k \geq 2$. Our strategy (eventually setting $s = 2n/3$) will be to extend a VMTL for $sK_3$ to a VMTL for the complete graph with the same $3s$ vertices. The next theorem calculates what the new magic constant would have to be.

**Theorem 10.** *Assume $s \geq 3$ and let $G_1 = (V_1, E_1)$ be isomorphic to $sK_3$. Let $G_2 = (V_2, E_2)$ be isomorphic to the complete graph $K_{3s}$, with $V_1 = V_2$ and $E_1 \subseteq E_2$. Let $\alpha_i$ be a VMTL of $G_i$ with magic constant $h_i$, for $i = 1, 2$. Assume that $\alpha_2$ agrees with $\alpha_1$ on $G_1$. Then, $h_2 = h_1 + \left(\frac{3s}{4}\right)(3s+1)^2 - \frac{21s}{2} - \frac{3}{2}$.*

**Proof.** By Proposition 1 part 4, $h_i = \frac{|E_i|^2}{|V_i|} + 2|E_i| + \frac{|E_i|}{|V_i|} + |V_i| + 1 - \frac{S_i}{|V_i|}$ where $S_i$ is the sum of vertex labels of $\alpha_i$. Since $\alpha_2$ agrees with $\alpha_1$ on vertices, it follows that $S_1 = S_2$. Since $V_1 = V_2$, we have:

$$h_2 - h_1 = \frac{|E_2|^2}{|V_2|} + 2|E_2| + \frac{|E_2|}{|V_2|} - \left(\frac{|E_1|^2}{|V_1|} + 2|E_1| + \frac{|E_1|}{|V_1|}\right).$$

Furthermore, since $|E_1| = |V_1| = 3s$ and $|E_2| = \frac{3s(3s-1)}{2}$, we have:

$$\begin{aligned} h_2 &= h_1 + \frac{3s(3s-1)^2}{4} + 3s(3s-1) + \frac{3s-1}{2} - (3s + 6s + 1) \\ &= h_1 + \frac{3s\left((3s-1)^2 + 12s\right)}{4} - 3s + \frac{3s-1}{2} - (9s+1) \\ &= h_1 + \frac{3s(3s+1)^2}{4} - \frac{21s}{2} - \frac{3}{2}. \quad \square \end{aligned}$$

Our tool for extending a VMTL for $sK_3$ to a VMTL for $K_{3s}$ is the following result of Shiu, Lam and Lee.

**Theorem 11** (*From [13]*)**.** *For integers $q$ and $s$, assume that $q \geq 3$, $s > 5$ and $s \not\equiv 0 \bmod 4$. Then the complete $s$-partite graph $K_{\underbrace{q, q, \ldots, q}_{s \text{ of them}}}$ is supermagic; i.e. it has a vertex-magic edge labeling.*

**Theorem 12.** *Let $n \geq 9$ be an odd multiple of 3, and set $s = 2n/3$. Assume that $\alpha_1$ is a VMTL for $sK_3$. Then $\alpha_1$ can be extended to a VMTL $\alpha_2$ for $K_{2n}$. Furthermore, letting $h_i$ be the magic constant of $\alpha_i$, we have $h_2 = h_1 + \left(\frac{3s}{4}\right)(3s+1)^2 - \frac{21s}{2} - \frac{3}{2}$.*

**Proof.** Let $G \cong K_{\underbrace{3, 3, \ldots, 3}_{s \text{ of them}}}$ be the complete $s$-partite graph with $3s$ vertices. First note that $s \not\equiv 0 \bmod 4$, since $n$ is odd. Furthermore, $s > 5$ since $n \geq 9$. Hence we may apply Theorem 11, with $q = 3$, to get a vertex-magic edge labeling $\sigma$ for $G$. Let $\bar{G}$ be the complement of $G$. Since $\bar{G} \cong sK_3$ it follows from our assumptions that $\bar{G}$ has a VMTL $\alpha_1$. Define $\alpha_2$ by setting $\alpha_2 = \alpha_1$ on $\bar{G}$, and set $\alpha_2 = \sigma + 6s$ on the edges of $G$. Note that $\alpha_2$ inherits the magic property from $\alpha_1$ and $\sigma$. Furthermore, $\bar{G}$ has $6s$ objects (vertices and edges) so that $\alpha_2$ is indeed a VMTL. The magic constant is calculated immediately by an application of Theorem 10. $\square$

Next we need VMTLs for $sK_3$, provided by Theorem 13, and then we put it all together in Theorem 14.

**Theorem 13** (*From [11]*)**.** *Assume $k \geq 1$ and let $s = 2 \cdot 3^k$. Then $sK_3$ has a VMTL with magic constant $h_1$ if and only if $(1/2)(15s + 4) \leq h_1 \leq (1/2)(21s + 2)$.*

**Theorem 14.** *Assume $k \geq 2$ and let $n = 3^k$. Then $K_{2n}$ has a VMTL with magic constant $h$ if and only if $h$ is an integer such that $2n^3 + (3/2)n \leq h \leq 2n^3 + 2n^2 + (n/2)$.*

**Proof.** First recall from the introduction that if $p$ is a positive integer, the feasible values for $K_p$ consist of the integer values of $h$ satisfying $(p/4)(p^2 + 3) \leq h \leq (p/4)(p+1)^2$. Setting $p = 2n$, this becomes the range of values described in the statement of the theorem. Since $2n^3 + (3/2)n$ and $2n^3 + 2n^2 + (n/2)$ are not integers, we will show that there is a VMTL for $K_{2n}$ with constant $h$ whenever:

$$2n^3 + (3/2)n + (1/2) \leq h \leq 2n^3 + 2n^2 + (n/2) - (1/2).$$

Since $n$ is odd, we can apply Corollary 2 to obtain a VMTL with magic constant $h$ whenever $2n^3 + 3n \leq h \leq 2n^3 + 2n^2 - n$. Thus, we will show that there is a VMTL with magic constant $h_2$ whenever $2n^3 + 2n^2 - n < h_2 \leq 2n^3 + 2n^2 + (n/2) - (1/2)$, and then duality (Proposition 2), applied to each of these new VMTLs automatically finishes the proof. Applying Theorem 12 on each VMTL from Theorem 13 yields a VMTL for $K_{2n}$ with magic constant $h_2$ for each $h_2$ that satisfies:

$$\frac{15s+4}{2} + \left(\frac{3s}{4}\right)(3s+1)^2 - \frac{21s}{2} - \frac{3}{2} \leq h_2 \leq \left(\frac{3s}{4}\right)(3s+1)^2 - \frac{1}{2}.$$

Since $3s = 2n$, this is equivalent to:

$$-2n + \frac{1}{2} + \frac{n}{2}(2n+1)^2 \leq h_2 \leq \frac{n}{2}(2n+1)^2 - \frac{1}{2}.$$

However, $-2n + (1/2) + (n/2)(2n+1)^2 = 2n^3 + 2n^2 - (3n/2) + (1/2) < 2n^3 + 2n^2 - n$, which is our required lower bound for magic constants, and $(n/2)(2n+1)^2 - (1/2) = 2n^3 + 2n^2 + (n/2) - (1/2)$, which is the desired upper bound for magic constants. The result follows. □

## 4. Concluding remarks

There are several conjectures and research problems along the same lines as Conjecture 1. Many of them are listed in Wallis' excellent book on magic graphs [16]. One in particular, due to Godbold and Slater [1] was actually a conjecture regarding *edge-magic total labelings* of graphs. However that conjecture is (trivially) equivalent to the following conjecture on VMTLs:

**Conjecture 2** (*Restated from [1]*)**.** *For each integer $p \geq 6$, and for each integer h such that $(1/2)(5p+3) \leq h \leq (1/2)(7p+3)$, there is a VMTL for the cycle $C_p$ with magic constant h.*

In [1], Conjecture 2 is verified for $6 \leq p \leq 10$. The authors also note that $C_5$ does not have a VMTL with a magic constant of 15 or 18, however they do provide VMTLs with the biggest and smallest feasible magic constant for each cycle. The even cycles and odd cycles are treated separately, and it is mentioned in [1] that it was "much more difficult" to find VMTLs for the even cycles than for the odd cycles. In [10], some progress is made on the spectrum of cycles by taking each VMTL for $C_p$ and using it to construct two VMTLs for $C_{np}$ provided that $n$ is odd—although $p$ can be even or odd.

In the same spirit as Conjectures 1 and 2, we make the following:

**Conjecture 3.** *There is an integer M such that each regular graph of order at least M has the property that it possesses VMTLs with each of its feasible values for magic constants.*

In [14], it is noted that a computer search has shown that the Petersen graph $P(7, 3)$ does not have a VMTL with magic constant 63—one of its feasible values. Therefore, if Conjecture 3 is true then $M > 14$, although $M$ may not be much larger than 14. Another example is the graph $4K_3$, which does not have a VMTL with its feasible value of 43 (see [11]). Three more examples are found in [4], where Gray and MacDougall note that none of the disjoint unions $2C_3 \cup C_5$, $C_4 \cup C_3$ or $C_4 \cup 3C_3$ possess strong VMTLs. They then ask if the disjoint unions $sC_3 \cup C_4$ ($s$ an odd integer) or $tC_3 \cup C_5$ ($t$ an even integer) could *ever* possess strong VMTLs. This question was answered in the affirmative in [6], where it is shown that, other than the three graphs just listed, they *always* possess them. Like in [1], it is noted in [4] that the even order graphs have "…always seemed much harder to deal with…" than the odd order graphs. Nevertheless, they provide many interesting new general methods which may help decide some of the questions regarding the spectrum of families of regular graphs.

Finally, we mention MacDougall's conjecture, which posits that each regular graph of degree at least 2, other than $2K_3$, possesses a VMTL. In fact, in an attempt to settle this conjecture, Gray provides [3] extremely general constructions providing VMTLs for large classes of regular graphs, including all Hamiltonian odd order regular graphs. His methods often involve assuming the existence of a VMTL of a regular graph of low degree in order to construct a VMTL for a graph of the same order, but higher degree. Therefore, some of his general constructions may well eventually lead to a solution of Conjecture 1.

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